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CAYLEY SNARKS AND ALMOST SIMPLE GROUPS

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A Cayley snark is a cubic Cayley graph which is not 3-edge-colourable. In the paper we discuss the problem of the existence of Cayley snarks. This problem is closely related to the problem of the existence of non-hamiltonian Cayley graphs and to the question whether every Cayley graph admits a nowhere-zero 4-flow.

So far, no Cayley snarks have been found. On the other hand, we prove that the smallest example of a Cayley snark, if it exists, comes either from a non-abelian simple group or from a group which has a single non-trivial proper normal subgroup. The subgroup must have index two and must be either non-abelian simple or the direct product of two isomorphic non-abelian simple groups.

1. Introduction

A *Cayley snark* is a cubic Cayley graph whose edges cannot be coloured by three colours in such a way that adjacent edges receive distinct colours. At present, no Cayley snarks are known, however, no argument has so far been given to exclude the existence of such graphs. The aim of the present paper is to show that in searching for a Cayley snark it is sufficient to restrict ourselves to a very limited class of finite groups—groups that are only one step away from being simple.

There are several reasons which make the problem of the existence of Cayley snarks interesting.

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First of all, every hamiltonian cubic graph is easily seen to be 3-edge-colourable. Therefore any example of a Cayley snark would be a non-hamiltonian Cayley graph. However, there is a well-known conjecture (of unknown origin) saying that every Cayley graph *is* hamiltonian. Although the conjecture appears to be supported by a host of papers showing that particular classes of groups give rise to hamiltonian Cayley graphs (see, e.g., [7, 12]), the evidence in favour of the conjecture is still very slight. This seems to have led Babai [3] to come up with the following counter-conjecture: *For some constant* $c>0$*, there are infinitely many vertex-transitive graphs, even Cayley graphs, without cycles of length greater than* $(1-c)n$*,* n *being the order of the graph.*

In either case, the solution of our Cayley snark problem would provide a valuable approximation of the possible solution of the above conjectures.

Closely related to the problem of the existence of hamiltonian cycles in Cayley graphs is another conjecture, due Alspach and Zhang (see Alspach, Liu, and Zhang [1]): *Every Cayley graph of valency at least two admits a nowhere-zero* 4*-flow.* Indeed, every hamiltonian graph is easily seen to have a nowhere-zero 4-flow. Moreover, with the help of a theorem of Jaeger on 4-flows in 4-edge-connected graphs [5, p. 134] the conjecture reduces to cubic Cayley graphs (see [1] for details). It is well-known, however, that a cubic graph admits a nowhere-zero 4-flow if and only if it is 3-edge-colourable [5, p. 135]. Hence, the conjecture of Alspach and Zhang is equivalent with the statement that there are no Cayley snarks.

In the background of all these questions there is the following general problem: *Which combinatorial properties of a graph are implied by its symmetry?* In particular, is there any relationship between the symmetry and edge-colourability of cubic graphs? The Petersen graph is a symmetrical cubic graph in a very strong sense, and yet it is not 3-edge-colourable. In fact, it is the only snark known so far to be vertex-transitive. However, it is not a Cayley graph. The exceptional position of the Petersen graph becomes even more surprising if we take into account that every snark can be expressed as a Schreier coset graph for some group and a suitable subgroup (see Proposition 1 in the next section). This observation leads us back to our original question rephrased as follows: *Does some snark admit a representation as a Cayley graph?*

As a partial answer to this question we prove the following result.

Main Theorem. *If there exists a Cayley snark, then there is a Cayley snark* $C(G;\{r,r^{-1},l\})$ *where* r *has odd order,* $l^2=1$*, and* $G=\langle r,l\rangle$ *is either a non-abelian simple group, or* G *has a unique non-trivial proper normal* *subgroup H which is either simple non-abelian or the direct product of two isomorphic non-abelian simple groups, and $|G\!:\!H|=2$.*

Let us remark that every non-abelian finite simple group can be generated by two suitable elements r and l where l is an involution (Malle, Saxl, and Weigel [8]).

2. Preliminaries

Graphs considered in this paper will be finite, non-trivial, and connected unless one can readily infer from the immediate context. Edges of our graphs are one of three types: *links*, *loops*, and *semiedges*. Multiple adjacencies are permitted. A link is incident with two vertices while a loop or a semiedge is incident with a single vertex. A link or a loop gives rise to two oppositely directed *arcs* that are *reverse* to each other. A semiedge incident with a vertex u gives rise to a single arc initiating at u that is reverse to itself. We use $L(x)$ to denote the reverse to an arc x.

Formally, a *graph* is a quadruple $K=(D,V;I,L)$ where $D=D(K)$ and $V = V(K)$ are disjoint non-empty finite sets, $I : D \to V$ is a surjective mapping, and L is an involutory permutation on D. The elements of D and V are *arcs* and *vertices*, respectively, I is the incidence function assigning to every arc its *initial vertex*, and L is the *arc-reversing involution*; the orbits of the group $\langle L\rangle$ on D are *edges* of K. It may happen that $L(x)=x$ for some arc x of K, and in this case the corresponding edge is called a *semiedge*. If $IL(x) = I(x)$ but $L(x) \neq x$, then the corresponding edge is a *loop*. The remaining edges are *links*. Our graphs are thus essentially the same as those in [11].

The usual graph-theoretical concepts such walks, cycles, connectedness, biparticity, etc., transfer to our graphs in the obvious way. In particular, the *valency* of a vertex v is the number of arcs having v as their initial vertex.

An *edge-colouring* of a graph $K=(D,V;I,L)$ is a mapping φ from the set of arcs of K to a set of colours F such that $\varphi(L(x))=\varphi(x)$. A colouring is said to be *regular* if, for each vertex v, the arcs emanating from v receive distinct colours. A regular colouring with $|F|=k$ is called a *k-edge-colouring.* Note that a graph which has a k-edge-colouring must be loopless.

A *snark* is a cubic (3-valent) graph *without loops and semiedges* which admits no 3-edge-colouring. More about the history, motivation, properties and constructions of snarks can be found in our paper [10].

Let G be a finite group with unit element e. Let H be a subgroup of G and let X be a generating set of G such that $X^{-1}=X$; we allow $e\in X$. The (left) *Schreier coset graph* $S(G/H;X)$ is a graph whose vertex-set is the set

C of left cosets of H in G and arc-set is $C\times X$; the initial vertex of an arc (gH,z) is gH, and the reverse to (gH,z) is (zgH,z^{-1}). If $H=\{e\}$, we may identify the cosets, i.e., the vertices of $S(G/H;X)$, with the elements of G. The graph $S(G/\{e\};X)$ is then usually denoted by $C(G;X)$ and called the (left) *Cayley graph* of G with respect to the generating set X.

It is well known that a Cayley graph is always vertex-transitive. This need not be true for Schreier coset graphs in general. Nevertheless, they are always k-valent for some value of k. If $S(G/H;X)$ is a cubic graph, that is, if $k=3$, then the generating set X must have one of two forms. Either $X=\{r,s,t\}$ where all the generators in X are involutions, or $X=\{r,r^{-1},l\}$ where l is an involution. Thus, according to its generating set, a cubic Schreier coset graph can be classified as either *balanced* or *unbalanced*, respectively.

In order to cover all the existing cases (and also to keep the cubicity), it is important to allow the possibility that in an unbalanced cubic Schreier coset graph r is an involution. In this case we assume that r and r^{-1} occur in X separately and these two occurrences are interchanged by L. This agreement guarantees that the inverse to an arc (gH,r) is always different from (gH,r).

Clearly, a balanced cubic Schreier coset graph is 3-edge-colourable. Therefore no snark can be represented in this way. So in what follows we concentrate on the study of unbalanced cubic Schreier coset graphs.

Let $K=C(G;\{r,r^{-1},l\})$ and $K'=C(G';\{r',(r')^{-1},l'\})$ be unbalanced cubic Cayley graphs such that $G'=G/N$ for some normal subgroup $N\trianglelefteq G$ and $r'=rN$ and $l'=lN$. Then the graph homomorphism induced by the projection sending any $g\in G$ to $g'=gN$ is a graph covering which will be called the *standard covering projection* of K onto the quotient K'. It is a consequence of the definition of a Cayley graph and the above agreement that the quotients of unbalanced cubic Cayley graphs by normal subgroups remain to be unbalanced cubic Cayley graphs. In particular, if $K=C(G;\{r,r^{-1},l\})$ and $N\trianglelefteq G$ contains l, the quotient $K'=C(G/N;\{rN,r^{-1}N,lN\})$ is nothing but a cycle with one semiedge attached to every vertex. Thus a standard quotient of an unbalanced cubic Cayley graph without semiedges may contain semiedges.

The following proposition emphasizes the importance of the family of unbalanced cubic Cayley graphs by showing that this family is, in a sense, universal for the family of bridgeless cubic graphs, and hence for snarks.

Proposition 1. *Every bridgeless cubic graph can be represented as an unbalanced Schreier coset graph.*

Proof. Let K be a bridgeless cubic graph of order n with vertices labelled $1,2,\dots,n$. We show that K is isomorphic to the Schreier coset graph $S(G/H;\{r,r^{-1},l\})$ for some $G=\langle r,l\rangle$ and $H\leq G$. By Petersen's theorem

[5, p. 36], K has a perfect matching M. The matching determines an involution $l \in S_n$ by the the following rule: $l(i) = j$ if ij is an edge in M. To define the other generator r we select an orientation of every cycle of the complementary 2-factor. The oriented 2-factor $F = K - M$ determines a permutation $r \in S_n$ by letting $r(i) = j$ if ij is an arc in F with initial vertex i and terminal vertex j. We set $G = \langle r, l \rangle$ and H the stabilizer of some vertex in K under the action of G on $V(K)$. It is a routine matter to verify that $K \cong S(G/H; \{r, r^{-1}, l\})$. ∎

3. Proof of the main result

Define a group G to be *almost simple* if it has a unique non-trivial proper normal subgroup H, and $|G : H| = 2$. The subgroup H will be called the *kernel of* G.

Proposition 2. *If G is a finite almost simple group with kernel H, then H is either simple or the direct product of two isomorphic simple groups.*

Proof. Clearly, H is a minimal characteristic subgroup of G. By [2, 8.2, p. 25], H is a direct product $H = T_1 \times T_2 \times \ldots \times T_n$, where $T_i \cong T$ is a simple group.

Consider an element $d \in G - H$. Let ν be the inner automorphism of G corresponding to d. Clearly $\nu(T_i)$ is a direct factor of H. Since $d^2 \in H$, and the inner automorphisms of G corresponding to elements of H leave each direct factor of H invariant, it follows that there is an involutory permutation $\sigma \in S_n$ such that $\nu(T_i) = T_{\sigma(i)}$. Let $U = \langle T_1, T_{\sigma(1)} \rangle$. Now $1 \neq U \trianglelefteq G$ and $U \leq H$, so $U = H$. Hence either $H = T_1$, or $H = T_1 \times T_2$ and $\nu(T_1) = T_2$. ∎

Our main result can now be stated as follows.

Theorem 3. *For every Cayley snark there exists a standard regular covering projection onto a Cayley snark $C(G; \{r, r^{-1}, l\})$ where $|r|$ is odd, $l^2 = 1$ and*

(1) *G is either non-abelian and simple, or*

(2) *G is almost simple with non-abelian kernel H, $G = H \rtimes \langle l \rangle$ and $|r| \geq 5$.*

Proof. Let $K_0 = C(G_0; X_0)$ be a Cayley snark. As we have already shown, X_0 is necessarily of the form $\{r_0, r_0^{-1}, l_0\}$, where l_0 is an involution. Moreover, it is easy to see that the order of r_0 must be odd. Otherwise, we can produce a 3-edge-colouring of K_0 by colouring the edges of the 2-factor corresponding to r_0 alternately with two colours and by assigning the third colour to the edges of the complementary 1-factor corresponding to l_0.

Let N_0 be a maximal proper normal subgroup of G_0 such that $r_0 \notin N_0$. We first prove that $l_0 \notin N_0$. Indeed, if $l_0 \in N_0$, then the quotient graph $K_0/N_0 = C(G_0/N_0; X_0/N_0)$ is isomorphic to a cycle C_m, $m \geq 2$, with one semiedge attached to each vertex (and, moreover, m divides $|r|$). Taking the semiedges in the cyclic order let us colour them consecutively by colours $1, 2, 3, 3, \ldots, 3, 3$. This assignment uniquely extends to a 3-colouring of K_0/N_0. By lifting this colouring to K_0 we see that K_0 is 3-edge-colourable, a contradiction. So $l_0 \notin N_0$.

Now set $G = G_0/N_0$ and $X = X_0/N_0$.

If G is simple, it clearly must be non-abelian; otherwise G is cyclic of prime order and the quotient K_0/N_0 is a cycle with one semiedge attached to every vertex. Consequently, K_0 is a 3-edge colourable cubic graph, a contradiction.

If G is not a simple group, let $N \trianglelefteq G$ be a non-trivial proper normal subgroup of G. Then $N = W/N_0$ for some subgroup W such that $N_0 \trianglelefteq W \trianglelefteq G_0$. By the choice of N_0, we have $r_0 \in W$. Set $r = r_0 N_0$ and $l = l_0 N_0$. Then $r \in N$ and $G = \langle r, l \rangle$. Moreover $N \neq G$ implying that $l \notin N$. Since $\langle l \rangle \cap N = 1$ and $G = \langle N, l \rangle$, we have $G = N \rtimes \langle l \rangle$.

We now claim that G is almost simple. Since $|G : N| = 2$, it remains to prove that N is the unique non-trivial proper normal subgroup of G. Let H be a non-trivial proper normal subgroup of G. Then $H = Z/N_0$ for a subgroup Z such that $N_0 \trianglelefteq Z \trianglelefteq G_0$. By the choice of N_0, $r \in H$ while $l \notin H$ (since $H \neq G$). It is now easy to see that H consists of all elements of G which can be expressed as words over r and l with even number of occurrences of l. As the same holds for N, we have $H = N$ as required. Thus we have proved that if $G = G_0/N_0$ is not simple, it must be almost simple with kernel H and $G = H \rtimes \langle l \rangle$.

We further prove that G cannot have abelian kernel. Assume that this is not the case. By Proposition 2, there are two cases to consider: either $H \cong \mathbb{Z}_p$ or $H \cong \mathbb{Z}_p \times \mathbb{Z}_p$ for some prime number p.

Let us first suppose that the kernel H of G is a cyclic group of prime order p. Clearly, $H = \langle r \rangle$. Since $G = \langle r \rangle \rtimes \langle l \rangle$, we have $lrl = r^e$ for some integer e, and since $l^2 = 1$, we conclude that $e^2 \equiv 1 \pmod{p}$. For p a prime, the only solutions of this congruence are $e \equiv 1 \pmod{p}$ and $e \equiv -1 \pmod{p}$. In either case, $C(G; X)$ is isomorphic to a p-prism and thus 3-edge-colourable, a contradiction.

The other possibility is $H = \mathbb{Z}_p \times \mathbb{Z}_p$. Consider the subgroup $S = \langle (rl)^2 \rangle \leq G$. Since $H \trianglelefteq G$ and $r \in H$, we see that $lrl \in H$. However, H is abelian so $(rl)^2 = r \cdot lrl = lrl \cdot r = (lr)^2$. It follows that $lSl = S$. Moreover, $rSr^{-1} = S$ because both r and $(rl)^2$ are elements of H. Thus $S \trianglelefteq G$ and, at the same time, $S \leq H \lhd G$. By minimality, $S = H$ or $S = 1$. In the former case, H is cyclic, which is absurd. In the latter case, $G = \langle r, l; r^p = l^2 = (rl)^2 = 1\rangle$ is a dihedral group of order $2p$, a contradiction.

Finally we show that $|r| \geq 5$ whenever G is almost simple. By way of contradiction, suppose that $|r| = 3$. Then $K' = S(G/\langle r\rangle; X)$ is a bipartite cubic graph such that $C(G;X)$ arises from K' by expanding each vertex of K' to a 3-cycle. It is well-known [5, p. 103] that each bipartite cubic graph has a 3-edge-colouring. This colouring easily extends to a 3-edge-colouring of $C(G;X)$, which is impossible. So $|r| \neq 3$ and since $|r|$ is odd, we have $|r| \geq 5$. This completes the proof. ∎

Remark. We have just seen that the Cayley graph $K = C(G;X)$ on an almost simple group G with $X = \{r, r^{-1}, l\}$ and $|r| = 3$ is 3-edge-colourable. In contrast to this, if G is simple and $|r| = 3$, we cannot say very much. Nevertheless, the resulting problem has an interesting reformulation. Indeed, the orientation of the 3-cycles in K corresponding to r induces local orientations around the vertices of the Schreier coset graph $K' = S(G/\langle r\rangle; X)$ and hence an embedding of K' on some orientable surface. This embedding is highly symmetrical—it is a regular map (see [9, 11] for the definition)—because G acts on the set of arcs of K' as a regular group of automorphisms of this embedding. In [9] we have shown that this consideration can be reversed. So we have the following problem.

Problem. Is the underlying graph of any orientable cubic regular map whose automorphism group is simple non-abelian necessarily 3-edge-colourable?

Note that such a regular map is dual to a regular triangulation of an orientable surface (with the same automorphism group). Therefore, this question is closely related to the conjecture of Grünbaum [6] which claims that the dual of every polyhedral triangulation of an orientable surface is 3-edge-colourable. In fact, Grünbaum's conjecture implies that there are no Cayley snarks.

We conclude this paper by deriving the main result of [1].

Recall that a group G is said to be *solvable* if it admits a subnormal series of subgroups $G = G_0 \trianglerighteq G_1 \trianglerighteq \ldots \trianglerighteq G_n = 1$ such that each factor G_i/G_{i+1} is abelian. It is easy to see that no group satisfying the condition (1) or condition (2) of Theorem 3 is solvable. Since any quotient of a solvable group is again solvable, it follows from Theorem 3 that a Cayley snark cannot arise from a solvable group.

Corollary 4. (Alspach et al.[1]) *Every cubic Cayley graph on a solvable group is 3-edge-colourable.* ∎

Although our main theorem generalizes this result, its proof is based on entirely different ideas and is independent from [1]. The advantage over the proof of Alspach et al. is that it avoids the use of the theorem of Castagna and Prins [4] claiming that every generalized Petersen graph is 3-edge-colourable (whose proof is quite lengthy).

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